Nonlinear Vibration of Simply Supported Angle Ply Laminated Plates

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The large amplitude oscillations of a simply supported plate, constructed of plies alternately oriented at $\pm\theta$ and unsymmetrically laminated about the midsurface are investigated. It is determined that the Berger approximation does not lead to a significant simplification of the problem and that the dynamic von Kármán equation must be used. It is concluded that for the single mode solution the bending stretching coupling terms do not directly enter the nonlinear term in the amplitude frequency equation. However since they do enter the linear frequency term these terms do affect the degree of nonlinearity of the system. For the multimode solution the bending stretching coupling terms do enter the nonlinear terms; however, it is determined that they may be neglected in special cases. The angle of the laminates can lead to varying types of ultraharmonic instabilities and a method of predicting these regions is given.

Nomenclature

A = amplitude
$A_{ij}A_{ij}^*\bar{A}_{ij}^*$, $B_{ij}B_{ij}^*\bar{B}_{ij}^*$, $D_{ij}D_{ij}^*\bar{D}_{ij}$, $F_{0i}G_{iij}G_{klmn}$, J_i = constants defined in text
$\bar{C}_{ij}$ = elastic constants
$F(x,y,t)$ = stress function
F_i = generalized force
N,M = stress resultants
a,b = plate length and width
h = plate thickness
r = a/b
t = time
u,v,w = displacements in the x, y, and z directions, respectively
β = nonlinearity parameter
γ = h/b
ϵ = strain
ϵ_0 = midplane strain
θ = angle of orientation of the individual plys
κ = plate curvature
ξ = generalized coordinate
ρ = mass density
σ = stress
$\tau,\bar{\tau}$ = nondimensional time
ω = frequency
ω_0 = linear frequency

Introduction

THE advent of fiber reinforced composites as structural elements has required the reanalysis of many structural problems. A series of recent papers has been concerned with the study of rectangular plates laminated of plys of fiber reinforced composite layers, and in particular the study of their linear vibrations. The static equations have been obtained in a stress function formulation by Stavsky[1] and the dynamic equations in a displacement formulation by Whitney and Leissa.[2] The latter authors also obtained solutions for the particular cases of an unsymmetric angle ply laminate where the fibers axes are oriented at $\pm\theta$ to a reference axis and are unsymmetrically laminated about the mid plane of the plate, and that of an unsymmetric cross ply laminate, where the laminates are alternately oriented at 0° and 90° to a reference axis. Both of these examples include bending stretching coupling. Their results indicate that as the number of plies increase the effect of the coupling terms decrease and the solution approaches the orthotropic plate solution.

There have been few investigations however of the large deflection vibration problem. Mayberry and Bert[3] presented some experimental results and compared them with orthotropic solutions. Recently Wu and Vinson[4,5] have looked at the problem of the large oscillations of an orthotropic plate including rotary inertia and shear deformation effects. Their results indicated that for length to thickness ratios greater than 40, the shear and rotary effects are small compared to the nonlinear effects.

The present note is an investigation of the nonlinear response of an unsymmetrically laminated angle ply composite. The boundary conditions will be assumed to be simply supported for the bending displacement and the edges are restrained against in plane displacement.

Equations of Motion

Two approaches have been used to analyze large amplitude vibrations of isotropic plates. Yamaki[6] used Galerkin's method on the dynamic von Kármán equations. On the other hand an approximation suggested by Berger[7] and extended by Wah[8] has been used to reduce the coupled fourth-order partial differential equations of von Kármán to a single fourth-order partial differential equation. The approximation appears to be accurate for moderately large deflections.[8]

1. von Kármán Equations

The static von Kármán type equations for unsymmetrically laminated composite plates have been presented by Stavsky.[9] The dynamic equations may be obtained by adding a transverse inertia term. If in plane, rotatory, and coupling inertia terms are not included the following equations are obtained

$$\rho h\ddot{w} + L_1 w - L_3 F = F_{,yy}w_{,xx} - 2F_{,xy}w_{,xy} + F_{,xx}w_{,yy} + p(x,y,t) \quad (1)$$

$$L_2 F + L_3 w = w_{,xy}{}^2 - w_{,xx}w_{,yy} \quad (2)$$

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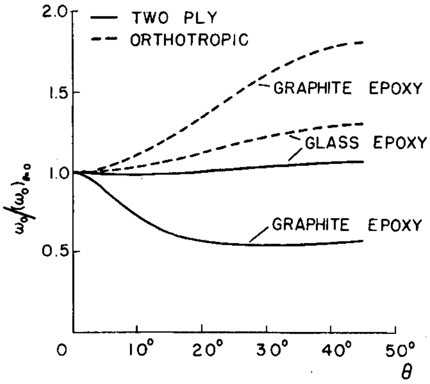

Fig. 1 Effect of angle of plies on the linear natural frequency for a square plate.

where

$$L_1 = D_{11}^* \frac{\partial^4}{\partial x^4} + 4D_{16}^* \frac{\partial^4}{\partial x^3 \partial y} + 2(D_{12}^* + 2D_{66}^*) \times \frac{\partial^4}{\partial x^2 \partial y^2} + 4D_{26}^* \frac{\partial^4}{\partial x \partial y^3} + D_{22}^* \frac{\partial^4}{\partial y^4}$$

$$L_2 = A_{22}^* \frac{\partial^4}{\partial x^4} - 2A_{26}^* \frac{\partial^4}{\partial x^3 \partial y} + (2A_{12}^* + A_{66}^*) \times \frac{\partial^4}{\partial x^2 \partial y^2} - 2A_{16}^* \frac{\partial^4}{\partial x \partial y^3} + A_{11}^* \frac{\partial^4}{\partial y^4} \quad (3)$$

$$L_3 = B_{21}^* \frac{\partial^4}{\partial x^4} + (2B_{26}^* - B_{61}^*) \frac{\partial^4}{\partial x^3 \partial y} + (B_{11}^* + B_{22}^* - 2B_{66}^*) \frac{\partial^4}{\partial x^2 \partial y^2} + (2B_{16}^* - B_{62}^*) \times \frac{\partial^4}{\partial x \partial y^3} + B_{12}^* \frac{\partial^4}{\partial y^4}$$

The stress function F is related to the stresses resultants by

$$F_{,yy} = N_x,\ F_{,xx} = N_y,\ F_{,xy} = -N_{xy} \quad (4)$$

The constants may be obtained from the following relations

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \epsilon_0 \\ \kappa \end{Bmatrix} \quad (5)$$

$$A_{ij}, B_{ij}, D_{ij} = \int_{-h/2}^{h/2} \bar{C}_{ij}(1, z, z^2) dz \quad (6)$$

where

$$\{\sigma\} = [\bar{C}]\{\epsilon\} \quad (7)$$

and

$$[A^*] = [A]^{-1}; \quad [B^*] = -[A]^{-1}[B]; \qquad [D^*] = [D] - [B][A]^{-1}[B] \quad (8)$$

The $[\bar{C}]$ matrix must be obtained from the elastic properties of the individual plies and their orientation. Transformations such as those given in Hearmon[10] must be used.

For the angle ply plate it is found that

$$B_{11} = B_{21} = B_{66} = B_{22} = 0, \quad A_{16} = A_{26} = 0 \qquad D_{16} = D_{26} = 0 \quad (9)$$

Equations (3) then become

$$L_1 = D_{11}^* \partial^4/\partial x^4 + 2(D_{12}^* + 2D_{66}^*)\partial^4/\partial x^4 \partial y^2 + D_{22}^* \partial^4/\partial y^4 \quad (10a)$$

$$L_2 = A_{22}^* \partial^4/\partial x^4 + (2A_{12}^* + A_{66}^*)\partial^4/\partial x^2 \partial y^2 + A_{11}^* \partial^4/\partial y^4 \quad (10b)$$

$$L_3 = (2B_{26}^* - B_{61}^*)\partial^4/\partial x^3 \partial y + (2B_{16}^* - B_{62}^*)\partial^4/\partial x \partial y^3 \quad (10c)$$

Equations (1) and (2) with the operators given by Eqs. (10) are then the differential equations for the large deformations of an unsymmetric angle ply laminate.

2. Berger Approximation

The approximation suggested by Berger involves neglecting certain terms in the strain energy expression. The strain energy for angle ply laminates may be written in the form

$$U = \frac{1}{2} \int_0^a \int_0^b \{D_{11} w^2_{,xx} + 2D_{12} w_{,xx} w_{,yy} + D_{22} w^2_{,yy} + 4D_{33} w^2_{,xy} - B_{16}[\epsilon_{xy_0}(2w_{,xx}) + 4\epsilon_{x_0} w_{,xy}] - B_{26}[\epsilon_{xy_0}(2w_{,yy}) + 4\epsilon_{y_0} w_{,xy}] + I_1^2 + I_2\} dx dy \quad (11)$$

where

$$\epsilon_{x_0} = u_{,x} + \tfrac{1}{2} w^2_{,x}, \quad \epsilon_{y_0} = v_{,y} + \tfrac{1}{2} w^2_{,y}$$

$$\epsilon_{xy_0} = \tfrac{1}{2}(u_{,y} + v_{,x}) + \tfrac{1}{2} w_{,x} w_{,y}, \quad I_1 = (A_{11}^{1/2} \epsilon_{x_0} + A_{22}^{1/2} \epsilon_{y_0})$$

$$I_2 = [2(A_{12} - A_{11}^{1/2} A_{22}^{1/2})]\epsilon_{x_0}\epsilon_{y_0} + A_{66}\epsilon_{xy_0}^2$$

Berger has suggested neglecting the I_2 term. If this is done for the orthotropic $[(B_{16} = B_{26} = 0)]$ case the following equations result from an application of the variational principle if longitudinal inertia terms are ignored.

$$\rho h w_{,tt} + D_{11} w_{,xxxx} + 2D_{12} w_{,xxyy} + D_{22} w_{,yyyy} + 4D_{66} w_{,xxyy} - [(I_1 w_{,x})_{,x} + (I_1 w_{,y})_{,y}] = p(x,y,t) \quad (12a)$$

and

$$I_{1,x} = 0, \quad I_{1,y} = 0 \quad (12b)$$

The latter two equations imply that I_1 is a constant and may be evaluated by integration. This reduces the problem to a single fourth-order partial differential equation in w.

In the case of angle ply laminates, however, this simplification cannot be made. The equations corresponding to Eqs. (12b) become

$$\begin{aligned} I_{1,x} + B_{16}(\tfrac{9}{2} w_{,xxy} + w_{,xyy}) + B_{26}(\tfrac{1}{2} w_{,yyy} + w_{,xyy}) &= 0 \\ I_{1,y} + B_{16}(\tfrac{1}{2} w_{,xxx} + w_{,xxy}) + B_{26}(\tfrac{9}{2} w_{,xyy} + w_{,xxy}) &= 0 \end{aligned} \quad (13)$$

It appears that no advantages of simplicity will arise from the Berger approximation in this case.

Boundary Conditions

If the plate is restrained in such a way that there is no in plane motion of the edges the following conditions must be satisfied[4]:

$$\int_0^a \left(\epsilon_{x_0} - \frac{1}{2} w_{,x}^2\right) dx = 0, \quad \int_0^b \left(\epsilon_{y_0} - \frac{1}{2} w_{,y}^2\right) dy = 0 \quad (14)$$

It has been shown by Chu and Herrmann[11] that the tangential

Table 1 Material properties

	Glass epoxy[12]	Boron epoxy[15]	Graphite epoxy[12]
E_{11}	7.5×10^6 psi	30×10^6 psi	30.0×10^6 psi
E_{22}	2.6×10^6 psi	3.0×10^6 psi	0.75×10^6 psi
ν_{12}	0.25	0.256	0.25
G_{12}	1.1×10^6 psi	1.2×10^6 psi	0.75×10^6 psi

displacements on the boundary are nonzero. Therefore the boundary conditions must permit tangential motion.

Solution of the Equations for a Single Assumed Mode

A single term Galerkin technique will be applied to Eqs. (1) and (2). The transverse displacement will be assumed to have the shape of the lowest linear isotropic mode

$$w(x,y,t) = b\xi(t)\sin(\pi x/a)\sin(\pi y/b) \tag{15}$$

This function is then substituted into Eq. (2) and a solution for F is sought. A homogeneous solution is

$$F_h(x,y,t) = C_1x^2 + C_2y^2 \tag{16}$$

and a particular solution is

$$F_p = C_3\cos(2\pi x/a) + C_4\cos(2\pi y/b) + C_5\cos(\pi x/a)\cos(\pi y/b) \tag{17}$$

where

$$C_3 = a^2\xi^2/32A_{22}^*$$
$$C_4 = b^2\xi^2/32r^2A_{11}^*$$
$$C_5 = \frac{(B_{61}^* - 2B_{26}^*)\pi^4 a + (B_{62}^* - 2B_{16}^*)\pi^4 a/r^2}{A_{22}^*\pi^4/r^4 + (2A_{12}^* + A_{66}^*)\pi^4/r^2 + A_{11}\pi^4}\cdot\xi \tag{18}$$
$$r = a/b$$

C_1 and C_2 can then be determined by the in plane boundary conditions given by Eqs. (14). If this is done

$$C_1 = (\pi^2r^2A_{12}^* - \pi^2A_{11}^*)/16(A_{12}^{*2} - A_{11}^*A_{22}^*)$$
$$C_2 = (\pi^2A_{12}^* - \pi^2r^2A_{22}^*)/16(A_{12}^{*2} - A_{11}^*A_{22}^*)$$

$F(x,y,t)$ may now be substituted into Eq. (1). Galerkin's method may be applied to this equation and if the following nondimensional quantities are noted we get

$$\xi_{,\tau\tau} + \frac{\pi^4\gamma^2}{r^4E_{11}}\left\{\bar{D}_{11}^* + 2(\bar{D}_{12}^* + 2\bar{D}_{66}^*)r^2 + \bar{D}_{22}^*r^4 + \frac{[(\bar{B}_{61} - 2\bar{B}_{26}) + (\bar{B}_{62} - 2\bar{B}_{16}^*)r^2]^2}{r^2[\bar{A}_{22}^* + (2\bar{A}_{12}^* + \bar{A}_{33}^*)r^2 + \bar{A}_{11}^*r^4]}\right\}\xi + \frac{\pi^4}{E_{11}8r^4}\times\left\{\frac{2}{\bar{A}_{11}^*} + \frac{2r^4}{\bar{A}_{22}^*} + \frac{(2\bar{A}_{12}^*r^2 - \bar{A}_{22}^* - \bar{A}_{11}^*r^4)}{\bar{A}_{12}^* - \bar{A}_{11}^*\bar{A}_{22}^*}\right\}\xi^3 = F(\tau) \tag{19}$$

where

$$\tau = (E_{11}/\rho)^{1/2}t/b, \quad \gamma = h/b$$
$$\bar{A}_{ij}^* = A_{ij}^*h; \quad \bar{B}_{ij}^* = B_{ij}^*/h^2; \quad \bar{D}_{ij}^* = D_{ij}^*/h^3$$

E_{11} is Young's molulus parallel to the fibers.

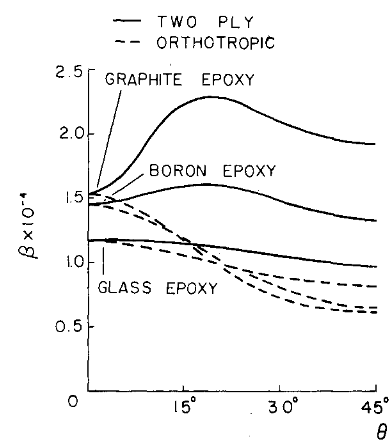

Fig. 2 Variation of nonlinearity parameter β with angle of plys for a square plate.

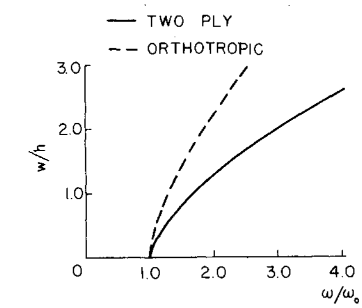

Fig. 3 Backbone curves for Graphite Epoxy for $\theta = 45°$, $r = 1.0$.

This equation is of the Duffing type and if the forcing is harmonic, Eq. (19) may be written

$$\xi_{,\tau\tau} + \omega_0^2\xi + G_{111}\xi^3 = P_0\cos\omega\tau \tag{20}$$

It is interesting to note that the nonlinear term depends on only the $\bar{A}_{ij}^*$ terms and not on the coupling terms $\bar{B}_{ij}^*$ which enter only the linear natural frequency term. Therefore the bending stretching coupling terms do not directly enter the nonlinear term.

Solution of the Duffing Equation

If an approximate solution to Eq. (20) is assumed in the form

$$\xi = A_1\cos\omega\tau$$

The method of harmonic balance leads to the following amplitude frequency relationship

$$\omega^2 = \omega_0^2 + \tfrac{3}{4}G_{111}A_1^2 - P_0/A_1 \tag{21}$$

It is convenient to rewrite this in the following fashion for the case $P_0 = 0$

$$(\omega/\omega_0)^2 = 1 + \beta A_1^2, \quad \beta = \tfrac{3}{4}G_{111}/\omega_0^2 \tag{22}$$

Equation (22) is called the backbone curve. The β term is essentially a measure of the degree of nonlinearity of the system. Therefore, even though the bending stretching coupling does not explicitly enter the nonlinear term, it does effect the degree of nonlinearity of the problem.

Numerical results have been calculated using the previously published elastic constants given in Table 1; γ was taken to be 0.01. The results are shown in Figs. 1–3. It is clear that for the two ply plate the bending stretching coupling introduces effects that cannot be determined by the orthotropic solution; however, as was pointed out earlier this effect is directly through the linear term. It appears that no general statements can be made about the effect of the angle of lamination on the nonlinearity parameter. However, it should be noticed that the more highly anisotropic laminates show the greatest variation between the anisotropic solutions and the orthotropic solutions.

Solution of the Equations for Several Assumed Modes

Earlier work on isotropic beams has shown[13] that often it is inadequate to consider one degree of freedom. Therefore the solution is assumed in the following form:

$$w(x,y,t) = b\sum_{m=1}^{j}\sum_{n=1}^{k}\xi_{mn}(\tau)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{23}$$

The case $j = k = 2$ will be considered. Following the same steps used in the single mode solution the Galerkin method yields the following set of four nonlinear algebraic equations:

$$\ddot{\xi}_1 + F_{01}\xi_1 + G_{111}\xi_1^3 + G_{112}\xi_1\xi_2^2 + G_{113}\xi_1\xi_3^2 + G_{114}\xi_1\xi_4^2 + G_{1234}\xi_2\xi_3\xi_4 + J_1\xi_2\xi_3 = F_1(\tau) \tag{24a}$$

Table 2 Summary of coefficients for Eqs. (24)

Theta coefficient	Graphite epoxy				Glass epoxy			
	0°	15°	30°	45°	0°	15°	30°	45°
$F_{01} \times 10^3$	0.9229	0.5842	0.5157	0.5340	1.734	1.734	1.782	1.837
$F_{02} \times 10^3$	1.501	1.395	1.980	2.929	7.897	8.269	9.348	10.74
$F_{03} \times 10^3$	13.37	6.556	4.083	2.929	16.01	14.55	12.34	10.74
$F_{04} \times 10^3$	14.76	9.347	8.252	8.554	27.74	27.74	28.51	29.37
G_{111}	18.86	17.41	14.49	13.74	27.06	26.40	24.89	24.06
G_{222}	25.93	28.42	42.80	82.53	126.5	128.1	138.3	16.92
G_{333}	292.8	249.1	151.8	82.53	307.8	283.7	225.4	16.92
G_{444}	301.7	278.6	231.9	219.8	433.0	422.5	398.2	384.09
G_{112}	39.30	36.13	31.63	39.81	74.57	74.21	73.93	78.92
G_{113}	76.58	71.94	58.05	39.81	103.3	100.4	90.67	78.92
G_{114}	54.21	58.60	57.56	54.54	91.91	92.63	92.04	90.81
G_{223}	68.47	92.56	124.3	171.3	190.3	208.0	237.6	251.7
G_{224}	97.59	110.4	123.4	167.0	321.2	323.7	334.2	377.4
G_{334}	587.7	484.4	266.7	167.0	644.2	596.8	480.2	377.4
G_{1234}	105.4	89.17	40.88	24.04	156.5	147.1	122.0	106.7
G_{4123}	103.6	85.80	32.71	53.25	142.2	130.2	100.1	82.02
$J_1 \times 10^3$	0.0	0.9513	−16.07	−2.387	0.0	−1.974	−8.106	−0.3145
$J_2 \times 10^3$	0.0	−28.13	−2.422	−3.259	0.0	15.50	19.88	−23.51
$J_3 \times 10^3$	0.0	−22.61	−28.85	3.259	0.0	−91.27	−93.92	−23.51
$J_4 \times 10^3$	0.0	−5.637	−3.475	−0.7236	0.0	−8.419	−8.230	−5.223

$$\ddot{\xi}_2 + F_{02}\xi_2 + G_{222}\xi_2^3 + G_{221}\xi_2\xi_1^2 + G_{223}\xi_2\xi_3^2 + G_{224}\xi_2\xi_4^2 + G_{2134}\xi_1\xi_3\xi_4 + J_2\xi_1\xi_3 = F_2(\tau) \quad (24b)$$

$$\ddot{\xi}_3 + F_{03}\xi_3 + G_{333}\xi_3^3 + G_{331}\xi_3\xi_1^2 + G_{332}\xi_3\xi_2^2 + G_{334}\xi_3\xi_4^2 + G_{3124}\xi_1\xi_2\xi_4 + J_3\xi_1\xi_2 = F_3(\tau) \quad (24c)$$

$$\ddot{\xi}_4 + F_{04}\xi_4 + G_{444}\xi_4^3 + G_{441}\xi_4\xi_1^2 + G_{442}\xi_4\xi_2^2 + G_{443}\xi_4\xi_3^2 + G_{4123}\xi_1\xi_2\xi_3 + J_4\xi_1^2 = F_4(\tau) \quad (24d)$$

where

$$\xi_1 = \xi_{11}, \quad \xi_2 = \xi_{12}, \quad \xi_3 = \xi_{21}, \quad \xi_4 = \xi_{22}$$

The coefficients $F_{01}\ldots F_{04}$, $G_{111}\ldots G_{444}$, $J_1\ldots J_4$ are given in the Appendix. It should be noted that the bending stretching coupling terms enter only the linear frequency terms, $F_{01}\ldots F_{04}$, and the terms J_1,J_2,J_3,J_4. A brief summary of all coefficients for various values of θ for a square plate for graphite epoxy and glass epoxy are listed in Table 2. The small absolute value of the J_i's compared to the other nonlinear coefficients should be noted. In resonance regions the terms involving the cubes of the generalized coordinates are obviously the most significant, therefore in many cases it will be practical to neglect the J_i's.

Symmetric Forcing

Consider the case of a load $\bar{P}_0 \cos\omega\tau$ applied at $(a/2, b/2)$ on a square plate. Then the right-hand sides of Eqs. (24) reduce to

$$F_1 = P_0 \cos\omega\tau, \quad F_2 = 0, \quad F_3 = 0, \quad F_4 = 0 \quad (25)$$

A possible solution is $\xi_1 \neq 0$, $\xi_2 = 0$, $\xi_3 = 0$ and if J_4 is neglected $\xi_4 = 0$. The response problem then reduces to Eq. (20) and the solution reduces to Eq. (21). Now the stability of this solution must be ascertained. Assuming a small pertubation, $\delta\xi_i$, of the steady-state value of each of the modes, and retaining only the first-order terms in the pertubations the following equations are obtained:

$$\frac{d^2\delta\xi_1}{d\bar{t}^2} + \frac{1}{4\omega^2}\left[F_{01} + \frac{3G_{111}A_1^2}{2} + \frac{3G_{111}A_1^2}{2}\cos\bar{t}\right]\delta\xi_1 = 0 \quad (26a)$$

$$\frac{d^2\delta\xi_i}{d\bar{t}^2} + \frac{1}{4\omega^2}\left[F_{0i} + \frac{G_{ii1}A_1^2}{2} + \frac{G_{ii1}A_1^2}{2}\cos\bar{t}\right]\delta\xi_i = 0 \qquad i = 2,3,4 \quad (26b)$$

where $\bar{t} = 2\omega\tau$.

The stability of these Mathieu equations can be examined using the well-known Mathieu stability diagram. Each equation will produce a region of the $A_1,\omega/\omega_0$ plane where the solution (21) and $\xi_2 = \xi_3 = \xi_4 = 0$ is unstable. The solutions that arise in these regions will be of the ultraharmonic type. The instability regions of ξ_2,ξ_3,ξ_4 are of particular interest. The solutions in these regions will be of the type

$$\xi_1 = A_1 \cos\omega\tau, \; \xi_i = A_i \cos n\omega\tau \qquad n = 1,2,3\ldots \quad (27)$$

where n corresponds to the number of the instability region in the Mathieu diagram. The solutions (27) will originate near the point

$$(\omega/\omega_0)^2 = F_{0i}/n^2F_{01} \quad (28)$$

Since the F_{0i} are dependent on the bending stretching terms and θ, widely different types of instabilities may be obtained for various plates.

Stability Regions

The standard Mathieu equation is of the form

$$\delta\ddot{\xi} + (\bar{\delta} + \bar{\epsilon}\cos\bar{t})\delta\xi = 0 \quad (29)$$

Where for the present problem

$$\bar{\delta} = (1/4\omega^2)(F_{0i} + G_{ii1}A_1^2/2)$$
$$\bar{\epsilon} = (1/4\omega^2)(G_{ii1}A_1^2/2) \quad (30)$$

If P_0 is small the response can be approximated by the backbone curve for $\omega/\omega_0 > 1.0$. Solving Eq. (22) for A_1^2 and substituting into Eq. (30) the following is obtained

$$\bar{\delta} = [(F_{0i}/4F_{01} - G_{ii1}/6G_{111})(\omega_0/\omega)^2 + G_{ii1}/6G_{111}]$$
$$\bar{\epsilon} = [-(G_{ii1}/6G_{111})(\omega_0/\omega)^2 + G_{ii1}/6G_{111}] \quad (31)$$

The exact relationships for the stability boundaries in terms of $\bar{\epsilon}$ and $\bar{\delta}$ are quite complicated. However, suitable linear approximations for the regions of interest may be constructed. The approximations used are

1st region

$$\bar{\epsilon} = 0.5 - 2\bar{\delta} \qquad \text{left-hand boundary} \quad (32a)$$

$$\bar{\epsilon} = 0.70 - 2.5\bar{\delta} \qquad \text{right-hand boundary} \quad (32b)$$

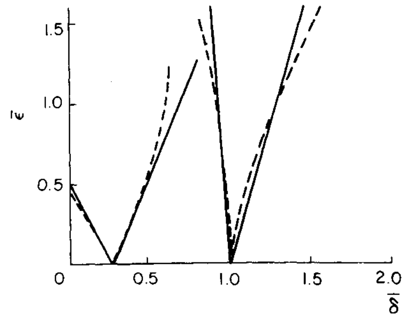

Fig. 4 Stability boundary approximations, —— approximations; – – exact solution.

2nd region

$$\bar{\epsilon} = 3.5 + 3.5\bar{\delta} \qquad \text{left-hand boundary} \qquad (32c)$$

$$\bar{\epsilon} = 13.3 - 13.3\bar{\delta} \qquad \text{right-hand boundary} \qquad (32d)$$

The exact boundaries and the approximations are shown in Fig. 4. These approximations have been chosen by trial and error to approximate the regions of interest. The approximations given by Stoker,[14] although rigorous, have the disadvantage of being valid only for small values of $\bar{\epsilon}$ in the first region and nonlinear in $\bar{\epsilon}$ for the second region. If more accurate answers are desired the higher order expansions (14) can be used; however, since the backbone is being used to approximate the forced response the accuracy would still be in question. The results presented in a later section confirm that the present assumptions lead to usable results.

Substituting Eqs. (31) into Eqs. (32) and solving for $(\omega/\omega_0)^2$ the following approximate stability boundaries are obtained. First region

$$\left(\frac{\omega}{\omega_0}\right)^2 = \frac{0.25G_{ii1}/G_{111} - 0.625F_{0i}/F_{01}}{-0.70 + 0.25\,G_{ii1}/G_{111}} \quad \text{left-hand boundary} \qquad (33a)$$

$$\left(\frac{\omega}{\omega_0}\right)^2 = \frac{-0.5G_{ii1}/G_{111} + 0.5F_{0i}/F_{01}}{0.5 - 0.5G_{ii1}/G_{111}} \quad \text{right-hand boundary} \qquad (33b)$$

Second region

$$\left(\frac{\omega}{\omega_0}\right)^2 = \frac{0.417\,G_{ii1}/G_{111} - 0.875\,F_{0i}/F_{01}}{-3.5 + 0.417\,G_{ii1}/G_{111}} \quad \text{left-hand boundary} \qquad (33c)$$

$$\left(\frac{\omega}{\omega_0}\right)^2 = \frac{-2.38\,G_{ii1}/G_{111} + 3.33\,F_{0i}/F_{01}}{13.3 - 2.38\,G_{ii1}/G_{111}} \quad \text{right-hand boundary} \qquad (33d)$$

If $(\omega/\omega_0)^2 < 0$ the boundary will not intercept the backbone

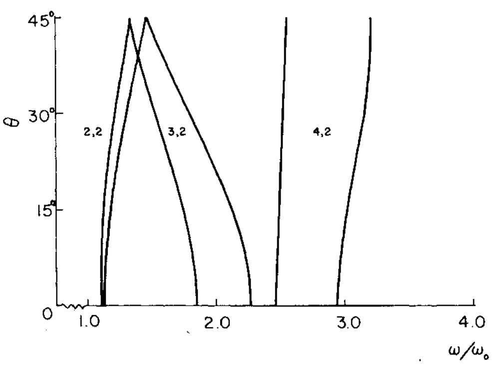

Fig. 6 Stability intercepts for glass epoxy, $r = 1.0$; n,m = region of instability for nth mode, mth ultraharmonic; flags are on stable side of boundary.

and if $0 \leq (\omega/\omega_0)^2 < 1$ Eq. (22) implies A_1 is imaginary since there is no backbone curve for $0 \leq \omega/\omega_0 < 1$

Stability intercepts as a function of θ for the three materials in Table 1 are presented in Figs. 5–7. The varying types of ultraharmonic instabilities and their dependence on θ are shown. Boundaries for both the orthotropic and two ply plates are shown in Fig. 5; however, because of the closeness of the boundaries the two ply case is not shown on the succeeding graphs. The boundaries in the two cases are close because the ratios F_{0i}/F_{01} and G_{ii1}/G_{111} are not very dependent on the $\bar{B}^*$ terms.

Examples of the full instability regions constructed from the exact Mathieu diagram for Graphite Epoxy for $\theta = 10°$ and $\theta = 40°$ are shown in Figs. 8 and 9. Note that there are slight decrepancies between the stability boundaries in Fig. 6 and Figs. 8 and 9 due to the non-zero value of P_0. However these errors are small indicating that the approximations are valid.

Experiments with isotropic beams[13] have shown that the narrow instability regions (small values of A_1) produce small amplitude instabilities which quickly damp out. However the wider regions orresponding to higher values of A_1 procduce instabilities that may have large amplitudes. For example, for the case $\theta = 40°$ neither the second order ultraharmonic of ξ_2, (2,2), nor the second order ultraharmonic of ξ_3, (3,2), will be significant. However the second order ultraharmonic of ξ_4, (4,2), will probably significantly affect the response for $\omega/\omega_0 > 2.5$. However for $\theta = 10°$ the first significant instability will be 2,1 for $\omega/\omega_0 > 2.25$.

Effect of the Nonlinear Terms due to Bending Stretching Coupling

The only term that has been neglected at this point is J_4. Therefore ξ_2 is not identically equal to zero and Eqs. (24a) and (24d) should be solved simultaneously. If $\xi_1 \simeq A_1 \cos\omega\tau$ the

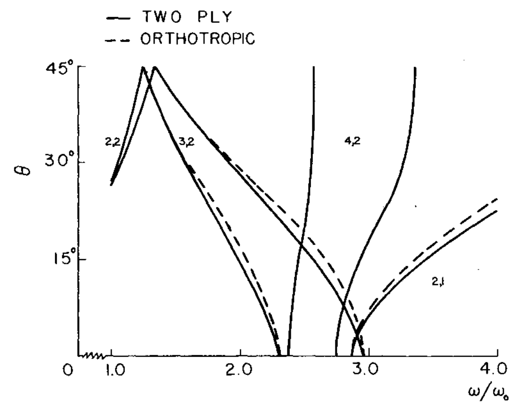

Fig. 5 Stability intercepts for boron epoxy, $r = 1.0$; n,m = region of instability for nth mode, mth ultraharmonic; flags are on stable side of boundary.

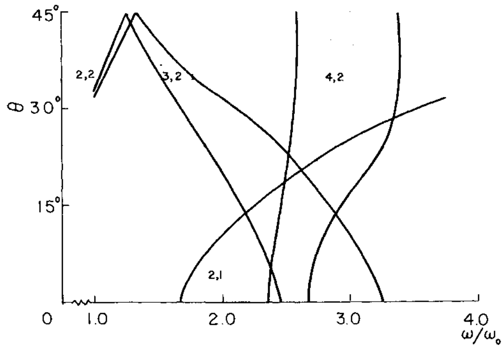

Fig. 7 Stability intercepts for graphite epoxy, $r = 1.0$; n,m = region of instability for nth mode, mth ultraharmonic; flags are on stable side of boundary.

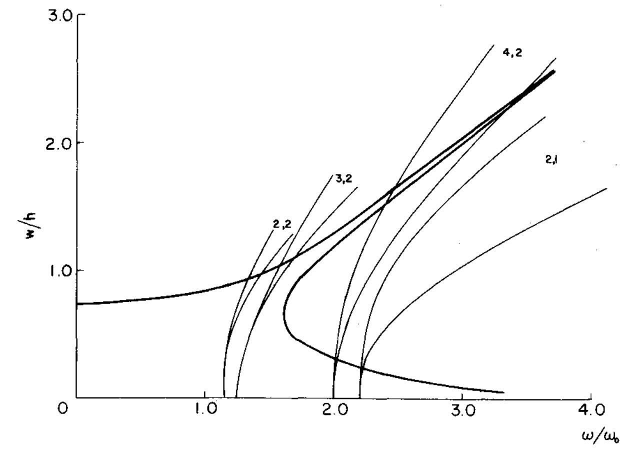

Fig. 8 Instability regions for graphite epoxy $\theta = 40°$, $r = 1.0$ n,m = region of instability for nth mode, mth ultraharmonic; flags are on stable side of boundary.

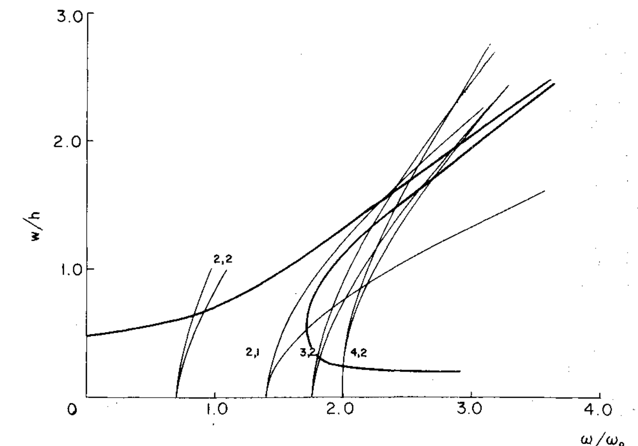

Fig. 9 Instability regions for graphite epoxy $\theta = 10°$, $r = 1.0$; n,m = region of instability for nth mode, mth ultraharmonic; flags are on stable side of boundary.

Eq. (24d) will behave as if it were forced by a function $0.5A_1^2(1 + \cos 2\omega\tau)$ which will produce a resonance in the region $\omega/\omega_0 = 0.5(F_{04}/F_{01})^{1/2}$. However, once the amplitude of ξ_4 becomes large the term $G_{441}\xi_1^2\xi_4$ will predominate over the term $J_4\xi_1^2$. This is basically the situation that arises when J_4 was neglected previously.

Conclusions

The analysis has indicated that for angle ply laminated plates it is possible to reduce the nonlinear partial differential equation to ordinary differential equations using Galerkin's method. If only one spatial mode is used the bending stretching terms do not directly enter the nonlinear terms, but do effect the linear frequency term. Thus, the angle of the ply laminates determines the degree of nonlinearity of the plate. Although the bending stretching terms do enter the nonlinear terms for the multispatial mode solution these terms are small. The type of instability response is dependent on the angle of the laminate but there are insignificant differences between the orthotropic case and the coupling case. It also appears that the Berger approximation does not lead to any appreciable simplification of the equations. However, in the cases where the orthotropic equations may be used without significant loss of accuracy the Berger approximation will provide a simpler way of obtaining adequate approximations of the ordinary differential equations.

Appendix

The coefficients in Eqs. (24) are as follows:

$$F_{01} = \frac{\pi^4\gamma^2}{E_{11}r^4}\left\{\bar{D}_{11}^* + 2(\bar{D}_{12}^* + 2\bar{D}_{66}^*)r^2 + \bar{D}_{22}^*r^4 + \frac{[(\bar{B}_{61}^* - 2\bar{B}_{26}^*) + (\bar{B}_{62}^* - 2\bar{B}_{16}^*)r^2]^2}{r^2(\bar{A}_{22}^* + (2\bar{A}_{12}^* + \bar{A}_{66}^*)r^2 + \bar{A}_{11}^*r^4)}\right\}$$

$$F_{02} = \frac{\pi^4\gamma^2}{E_{11}r^4}\left\{\bar{D}_{11}^* + 8(\bar{D}_{12}^* + 2\bar{D}_{66}^*)r^2 + 16\bar{D}_{22}^*r^4 + \frac{[2(\bar{B}_{61}^* - 2\bar{B}_{26}^*) + 8(\bar{B}_{62}^* - 2\bar{B}_{16}^*)r^2]^2}{r^2[\bar{A}_{22}^* + 4(2\bar{A}_{12}^* + \bar{A}_{66}^*)r^2 + 16\bar{A}_{11}^*r^4]}\right\}$$

$$F_{03} = \frac{\pi^4\gamma^2}{E_{11}r^4}\left\{16\bar{D}_{11}^* + 8(\bar{D}_{12}^* + 2\bar{D}_{66}^*)r^2 + \bar{D}_{22}^*r^4 + \frac{[8(\bar{B}_{61}^* - 2\bar{B}_{26}^*) - 2(\bar{B}_{62}^* - 2\bar{B}_{16}^*)r^2]^2}{r^2[16\bar{A}_{22}^* + 4(2\bar{A}_{12}^* + \bar{A}_{66}^*)r^2 + \bar{A}_{11}^*r^4]}\right\}$$

$$F_{04} = 16F_{01}$$

$$G_{111} = \frac{\pi^4}{E_{11}r^4}\left\{\frac{1}{16\bar{A}_{11}^*} + \frac{r^4}{16\bar{A}_{22}^*} + \frac{\bar{A}_{11}^*r^4 - 2r^2\bar{A}_{12}^* + \bar{A}_{22}^*}{8(\bar{A}_{11}^*\bar{A}_{22}^* - \bar{A}_{12}^{*2})}\right\}$$

$$G_{112} = \frac{\pi^4}{E_{11}r^4}\left\{\frac{1}{4\bar{A}_{11}^*} + \frac{r^4}{4\bar{A}_{22}^*} + \frac{81}{16H_{21}} + \frac{1}{16H_{23}} + \frac{4\bar{A}_{11}^*r^4 - 5\bar{A}_{12}^*r^2 + \bar{A}_{22}^*}{8(\bar{A}_{11}^*\bar{A}_{22}^* - \bar{A}_{12}^{*2})}\right\}$$

$$G_{113} = \frac{\pi^4}{E_{11}r^4}\left\{\frac{1}{4\bar{A}_{11}^*} + \frac{r^4}{4\bar{A}_{22}^*} + \frac{81}{16H_{12}} + \frac{1}{16H_{32}} + \frac{\bar{A}_{11}^*r^4 - 5\bar{A}_{12}^* + 4\bar{A}_{22}^*}{8(\bar{A}_{11}^*\bar{A}_{22}^* - \bar{A}_{12}^{*2})}\right\}$$

$$G_{114} = \frac{\pi^4}{E_{11}r^4}\left\{\frac{16}{H_{13}} + \frac{16}{H_{31}} + \frac{(\bar{A}_{11}^*r^4 - 2\bar{A}_{12}^*r^2 + \bar{A}_{11}^*)}{2(\bar{A}_{11}^*\bar{A}_{22}^* - \bar{A}_{12}^{*2})}\right\}$$

$$G_{1234} = \frac{\pi^4}{E_{11}r^4}\left\{\frac{1}{\bar{A}_{11}^*} + \frac{r^4}{\bar{A}_{22}^*} + \frac{25}{H_{13}} + \frac{25}{H_{31}}\right\}$$

$$J_1 = (9\pi^4\gamma/4E_{11}r^5)\{[2(\bar{B}_{61}^* - 2\bar{B}_{16}^*) + 8(\bar{B}_{62}^* - 2\bar{B}_{16}^*)r^2]/H_{12} + [8(\bar{B}_{61}^* - 2\bar{B}_{26}^*) + 2(\bar{B}_{62}^* - 2\bar{B}_{16}^*)r^2]/H_{21} - \tfrac{4}{3}[\bar{B}_{61}^* - 2\bar{B}_{26}^* + (\bar{B}_{62}^* - 2\bar{B}_{16}^*)r^2]/H_{11}\}$$

$$G_{222} = \frac{\pi^4}{E_{11}r^4}\left\{\frac{1}{16\bar{A}_{11}^*} + \frac{r^4}{\bar{A}_{22}^*} + \frac{16\bar{A}_{11}^*r^4 - 5\bar{A}_{12}^*r^2 + \bar{A}_{22}^*}{8(\bar{A}_{11}^*\bar{A}_{22}^* - \bar{A}_{12}^{*2})}\right\}$$

$$G_{223} = \frac{\pi^4}{E_{11}r^4}\left\{\frac{27}{4H_{11}} + \frac{625}{16H_{13}} + \frac{625}{16H_{31}} + \frac{27}{4H_{33}} + \frac{4\bar{A}_{11}^*r^4 - 17\bar{A}_{12}^*r^2 + 4\bar{A}_{22}^*}{8(\bar{A}_{11}^*\bar{A}_{22}^* - \bar{A}_{12}^{*2})}\right\}$$

$$G_{224} = \frac{\pi^4}{E_{11}r^4}\left\{\frac{1}{4\bar{A}_{11}^*} + \frac{4r^2}{\bar{A}_{22}^*} + \frac{81}{H_{14}} + \frac{4\bar{A}_{11}^*r^4 - 5\bar{A}_{12}^*r^2 + \bar{A}_{22}^*}{2(\bar{A}_{11}^*\bar{A}_{22}^* - \bar{A}_{12}^{*2})}\right\}$$

$$J_2 = 9\pi^4\gamma/(4E_{11}r^5)\{[8(\bar{B}_{61}^* - 2\bar{B}_{26}^*) + 2(\bar{B}_{62}^* - 2\bar{B}_{16}^*)r^2]/H_{21} - [2(\bar{B}_{61}^* - 2\bar{B}_{62}^*) + 8(\bar{B}_{62}^* - 2\bar{B}_{26}^*)r^2]/H_{12} - [\bar{B}_{61}^* - 2\bar{B}_{26}^* + (\bar{B}_{62}^* - 2\bar{B}_{16}^*)r^2]/H_{11}\}$$

$$G_{333} = \frac{\pi^4}{E_{11}r^4}\left\{\frac{1}{\bar{A}_{11}^*} + \frac{r^4}{16\bar{A}_{22}^*} + \frac{(\bar{A}_{11}^*r^4 - 5\bar{A}_{12}^*r^2 + 16\bar{A}_{22}^*)}{8(\bar{A}_{11}^*\bar{A}_{22}^* - \bar{A}_{12}^{*2})}\right\}$$

$$G_{334} = \frac{\pi^4}{E_{11}r^4}\left\{\frac{4}{\bar{A}_{11}^*} + \frac{r^4}{4\bar{A}_{22}^*} + \frac{81}{H_{41}} + \frac{\bar{A}_{11}^*r^4 - 5\bar{A}_{12}^*r^2 + 4\bar{A}_{22}^*}{8(\bar{A}_{11}^*\bar{A}_{22}^* - \bar{A}_{12}^*)}\right\}$$

$$J_3 = \tfrac{9}{4}(\pi^4\gamma/E_{11}r^5)\{[2(\bar{B}_{61}^* - 2\bar{B}_{26}^*) + 8(\bar{B}_{62}^* - 2\bar{B}_{16}^*)r^2]/H_{12} - [8(\bar{B}_{61}^* - 2\bar{B}_{26}^*) + 2(\bar{B}_{62}^* - 2\bar{B}_{16}^*)r^2]/H_{21} - [\bar{B}_{61}^* - 2\bar{B}_{26}^* + (\bar{B}_{62}^* - 2\bar{B}_{16}^*)r^2]/H_{11}\}$$

$$G_{444} = 16G_{111}$$

$$G_{4123} = \frac{\pi^4}{r^4E_{11}}\left\{\frac{1}{\bar{A}_{11}^*} + \frac{r_4}{\bar{A}_{22}^*} + \frac{25}{H_{13}} + \frac{25}{H_{31}} - \frac{3}{2H_{11}}\right\}$$

$$J_4 = -(\pi^4\gamma/2E_{11}r^5H_{11})[\bar{B}_{61}^* - 2\bar{B}_{26}^* + (\bar{B}_{62}^* - 2\bar{B}_{16}^*)r^2]$$

$$G_{221} = G_{112}, \quad G_{443} = G_{334}, \quad G_{331} = G_{113}, \quad G_{3124} = G_{2134} = G_{1234}$$

$$G_{441} = G_{114}, \quad G_{332} = G_{223}, \quad G_{442} = G_{224}$$

$$H_{mn} = (1/r^4)[m^4\bar{A}_{22}^* + m^2n^2(2\bar{A}_{12}^* + \bar{A}_{66}^*)r^2 + n^4\bar{A}_{11}^*]$$

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